STICHTING
MATHEMATISCH CENTRUM
2e BOERHAAVESTRAAT 49
AMSTERDAM

SP 108

(S 401)

W. Molenaar

How to poison Poisson (when approximating binomial tails)

How to poison Poisson (when approximating binomial tails) *

by W. MOLENAAR

Summary In these days some sportsmen are given dangerous drugs in order to stimulate them into still higher efforts. The present paper wants to add some poison to the classical approximation of a binomial by a Poisson distribution. The relatively harmless drug of a little extra calculation shall be seen to result in a much better accuracy.

This paper reports both on theoretical considerations, following from series expansions of Poisson-type parameters, and on extensive numerical investigations of accuracy by means of an Electrologica X8 computer. The main conclusion is that the probability of at most k successes in n Bernoulli trials with success probability $p \leqslant .5$ can be very closely approximated by the probability of at most k events in a Poisson distribution with expectation not np but

$$\zeta = \frac{(12-2p)n-7k}{(12-8p)n-k+k/n}\,np.$$

In the case $p > .5$ one should make $p < .5$ by a reversal, i.e. the interchanging of successes and failures, before the application of a Poisson-type approximation (with one exception mentioned in section 4). For the probability of at least j successes one should approximate the complementary probability of at most $j-1$ successes.

After an introductory section 1 and a discussion of measures of accuracy in section 2, the existing Poisson-type approximations are treated in section 3. The fourth section introduces a new parameter and discusses the advantages of reversal in other situations than $p > .5$. In section 5 a series expansion for the exact Poisson parameter is derived. Two new parameters, among which is ζ, are introduced in section 6, where the series expansions for parameters given in table 4 lead to conclusions about the various approximations, summarized in table 6. The final section 7 gives some numerical results, which confirm to a large extent the conclusions from the asymptotic expansions.

1. Introduction

Throughout this paper the random variable $\underline{x}$ denotes the number of successes in n independent trials with probability p of success. We introduce the notations

$$\begin{aligned} \mu &= np, \\ q &= 1-p, \\ b_j &= P\{\underline{x}=j\} = \binom{n}{j} p^j q^{n-j}, \\ B_k &= P\{\underline{x} \leqslant k\} = \sum_{j=0}^{k} b_j. \end{aligned} \tag{1.1}$$

The binomial distribution is most frequently used for the calculation of probabilities of exceedance in hypothesis testing, and for the determination of confidence bounds

* Report S 401 (SP 108), Statistical Department, Mathematical Centre, Amsterdam.

for a population proportion from a sample. There exist tables of B_k such as ORDNANCE CORPS (1952) for $n = 1$ (1) 150, $p = .01$ (.01) .50, and WEINTRAUB (1963) for small values of p. Even within the ranges of these tables approximations are often used, the tables being too bulky to be always available and interpolation being difficult.

Approximation by a normal distribution is most attractive, as anyone possesses the necessary tables. However, the accuracy of the classical normal approximation is sometimes poor, and more accurate normal approximations usually require substantial calculations. The present paper deals only with Poisson approximations. Some of them are accurate even near $p = .5$. When no cumulative Poisson table like GENERAL ELECTRIC COMPANY (1962) or MOLINA (1945) is available, the necessary values may be found from a chi-square table like table 7 of PEARSON and HARTLEY (1954) or from a nomogram (VERENIGING VOOR STATISTIEK nomogram 8.1).

Throughout this paper we give approximations for $B_k = P\{\underline{x} \leqslant k\}$, and with one exception in section 4 we shall assume that $p \leqslant .5$. For the approximation of $P\{\underline{x} \geqslant j\}$ with $p \leqslant .5$ one approximates B_k for $k = j-1$ and subtracts the result from 1. Approximation of B_k for $p > .5$ takes place by substitution of $1-p$ for p and of $n-k-1$ for k in the given formulae followed by subtraction from 1: at most k successes means at least $n-k$ failures. Similarly $P\{\underline{x} \geqslant j\}$ for $p > .5$ is found by replacing p by $1-p$ and k by $n-j$, without subtraction. The explicit statement of these simple rules may avoid misunderstandings.

We call B_k a left hand tail if it does not exceed .5; if it does, then $1-B_k = P\{\underline{x} \geqslant k+1\}$ is called right hand tail. Thus our binomial tail min $(B_k, 1-B_k)$ is never larger than .5.

When comparing approximations we shall work with the exact cumulative Poisson values, disregarding possible interpolation errors which may arise if the expectation lies between two values for which the distribution is tabled.

Some personal taste enters in the evaluation of the computational effort involved in a certain approximation. We take the view that it is desirable to avoid the use of special tables, such as 2 arcsin $\sqrt{x}$, the correction $\gamma(y,p)$ of BOLSHEV (1961) or the integral $I_p(2/3, 2/3)$ of BORGES (1969). Furthermore, addition of a not too simple correction term to a cumulative Poisson distribution function with a simple parameter is believed to be more laborious than directly looking up in a table the cumulative Poisson for a somewhat more complicated parameter.

It is common knowledge that the Poisson distribution with expectation $\mu = np$ approximates the binomial distribution for small p. However, both tails of the binomial are rather seriously overestimated unless p is very small indeed, as is illustrated in table 1.

The examples of table 1 have been selected to show that the approximated probabilities can be larger than some customary significance level while the actual probabilities are not. As is well known, this makes the Poisson approximation conservative in the sense that in hypothesis testing some significant results are not recognized as such, but no nonsignificant result is ever called significant.

Table 1

n	p	Binomial tail	Poisson (μ) approximation
5	.05	$P\{x \geqslant 2\} = .023$	.026
30	.1	$P\{x = 0\} = .042$	.050
40	.1	$P\{x \geqslant 8\} = .042$	.051
40	.2	$P\{x \geqslant 13\} = .043$	.064
70	.1	$P\{x \geqslant 12\} = .044$	.053
100	.1	$P\{x \leqslant 4\} = .024$	.029
100	.2	$P\{x \leqslant 13\} = .047$	.066
125	.05	$P\{x \geqslant 2\} = .048$	.052

Some textbooks qualify the Poisson approximation as good when n is large and p is small. Actually the value of n has little influence. HALD (1952) says: "It is generally considered justifiable to apply the Poisson distribution as approximation to the binomial distribution when $p < .1$". The two examples with $p = .05$ in table 1 exhibit relative errors of 17 and 9 percent, and show that even for such a small p correctness of two significant figures is far to seek.

2. Measures of accuracy

There are various ways to evaluate the accuracy of an approximation. Suppose each binomial term b_j is approximated by a_j. Let us put $A_k = \sum_{j=0}^{k} a_j$. Some criteria of accuracy found in the literature, giving one value for each binomial distribution, are

$$\sum_j |a_j - b_j| \qquad \text{(PROHOROV, 1953; LECAM, 1960)}$$

$$\max_k |A_k - B_k| \qquad \text{(HODGES and LECAM, 1960)} \qquad (2.1)$$

$$\max_{h,k} |(A_k - A_h) - (B_k - B_h)| \qquad \text{(RAFF, 1956)}$$

Only RAFF's paper deals specifically with the numerical accuracy of approximations, the other references being of a more theoretical character. A criterion of FREEMAN and TUKEY (1950), based on the normal deviates corresponding to A_k and B_k, is not suitable for us, as we work with some very skew distributions. All criteria in (2.1) concentrate the attention on the approximation of the middle part of the distribution, as the values for which the maximal absolute error is attained will rarely lie in the far tails. But practical applications of the binomial distribution will almost always concern these tails, either for hypothesis testing or for confidence bounds. Accurate approximation near the customary significance levels is more important than a low absolute error in the middle part of the distribution.

In this study we shall not try to judge approximations of a binomial distribution according to one criterion. We shall consider the relative tail error, defined as

$$E_k = 100\,\frac{A_k - B_k}{B_k} \quad \text{if} \quad B_k \leqslant .5; \qquad E_k = 100\,\frac{B_k - A_k}{1 - B_k} \text{ otherwise;} \qquad (2.2)$$

for all k such that $.001 < B_k < .999$. Approximations with a low E_k in the regions $.005 \leqslant B_k \leqslant .05$ and $.95 \leqslant B_k \leqslant .995$ are preferred, unless their performance in the middle region $.05 < B_k < .95$ is extremely bad.

3. Better approximations previously published

In section 1 the classical Poisson approximation

$$B_k \approx P_k(\mu) = \sum_{j=0}^{k} \frac{\mu^j}{j!} e^{-\mu} \qquad (\mu = np) \tag{3.1}$$

was found to overestimate both binomial tails even for relatively small p. We may ask what better parameter than μ a Poisson distribution should have, or what other correction should be applied, in order to obtain a more satisfactory approximation.

The earliest answer is given by the Gram-Charlier expansion (see e.g. RAFF (1956)):

$$B_k \approx \sum_{j=0}^{k} \frac{\mu^j}{j!} e^{-\mu} + \frac{(k-\mu)p}{2} \frac{\mu^k}{k!} e^{-\mu}. \tag{3.2}$$

The well known equality of binomial tail and incomplete beta integral, and also of Poisson tail and incomplete gamma integral (both specified in section 5) have been combined by WISE (1946, 1950) and by BOLSHEV and others (1961) with series expansion techniques, with the results

$$B_k \approx \sum_{j=0}^{k} \frac{\beta^j}{j!} e^{-\beta} - \frac{\beta(2\beta^2 - k\beta - k^2 - 2k)}{6(2n-k)^2} \frac{\beta^k}{k!} e^{-\beta}, \tag{3.3}$$

where $\beta = (2n-k)p/(2-p)$ (BOLSHEV c.s.) and

$$B_k \approx \sum_{j=0}^{k} \frac{\gamma^j}{j!} e^{-\gamma} + \frac{k+2+\gamma}{24} (\log q)^2 \frac{\gamma^{k-1}}{(k-1)!} e^{-\gamma}, \tag{3.4}$$

where $\gamma = -(n - \frac{1}{2}k) \log q$ (WISE). The approximations are remarkably accurate: e.g. for $n = 100$ and $p = .2$ the relative error in binomial tails of at least .001 is less than $\frac{1}{4}$ percent for (3.3) and less than 1 percent for (3.4). But the laborious calculations make them unattractive.

The use of (3.3) and (3.4) without their correction terms boils down to the use of β or γ, which are simple functions of n, p and k, instead of μ when looking up the probability of at most k events in a Poisson table. We shall consider

$$B_k \approx P_k(\beta) = \sum_{j=0}^{k} \frac{\beta^j}{j!} e^{-\beta} \qquad \text{where} \quad \beta = \frac{(2n-k)p}{2-p}; \tag{3.5}$$

$$B_k \approx P_k(\gamma) = \sum_{j=0}^{k} \frac{\gamma^j}{j!} e^{-\gamma} \qquad \text{where} \quad \gamma = -(n-\tfrac{1}{2}k)\log q; \tag{3.6}$$

as well as (3.2). The latter asks for somewhat more computational effort, including the use of a table of individual Poisson terms. Roughly speaking (3.2), (3.5) and (3.6) achieve the same order of accuracy, which is substantially higher than for the classical (3.1).

This is made plausible for BOLSHEV's (3.5) if we rewrite β as

$$\beta = (2n-k)p/(2-p) = \mu-(k-\mu)p/(2-p). \tag{3.7}$$

For left hand tails and $p \leqslant .5$, k is less than μ, thus β is larger than μ and the left Poisson tail is decreased when its parameter μ is replaced by β; the correction is stronger if p is larger. For right hand tails k is generally larger than μ, which implies $\beta < \mu$, the distribution function increases and the (so far overestimated) tail decreases.

The same argument holds for the Gram-Charlier approximation (3.2): $k-\mu$ being negative in the left hand tail and positive in the other, the overestimation by $P_k(\mu)$ is somewhat compensated; the compensation increases with p as does the error of $P_k(\mu)$.

Wise's parameter γ is exact for $k = 0$, as $e^{-\gamma} = q^n$ in this case. But (3.6) becomes inaccurate for large k, especially if p is far from zero.

In table 2 a few examples are found. As before, $P_k(\xi)$ denotes the probability of not exceeding k in a Poisson distribution with expectation ξ. To the six Poisson ap-

Table 2. Relative errors in percentages of the binomial tail, cf. (2.2)

n	p	k	B_k	tail	$P_k(\mu)$ (3.1)	$\Phi(u)$ (3.8)	$P_k(\beta)$ (3.5)	$P_k(\gamma)$ (3.6)	G.C. (3.2)	(3.3)	(3.4)
100	.1	3	.0078	.0078	+ 31.90	+93.07	+ .56	− .15	− 1.90	− .00	− .00
100	.1	5	.0576	.0576	+ 16.52	+16.03	+ .32	− .32	+ .09	+ .00	− .00
100	.1	6	.1172	.1172	+ 11.08	+ 3.86	+ .22	− .24	+ .32	+ .00	− .00
100	.1	13	.8761	.1239	+ 9.41	− 1.78	+ .34	+ .85	+ .58	+ .00	− .00
100	.1	15	.9601	.0399	+ 22.19	−16.33	+ .76	+ 1.45	+ .43	+ .00	− .01
100	.1	17	.9900	.0100	+ 42.67	−37.95	+ 1.40	+ 2.27	− 1.97	− .00	− .02
100	.3	19	.0089	.0089	+146.12	+23.47	+ 7.88	− 7.12	−19.87	− .11	− .24
100	.3	22	.0479	.0479	+ 68.32	+ 6.24	+ 4.57	− 7.00	+ 2.83	+ .01	− .22
100	.3	24	.1136	.1136	+ 38.45	+ 1.29	+ 2.83	− 6.64	+ 4.70	+ .03	− .19
100	.3	35	.8839	.1161	+ 35.58	− .90	+ 3.71	+ 14.25	+ 6.31	+ .06	− .51
100	.3	37	.9470	.0530	+ 67.80	− 4.13	+ 6.48	+ 19.78	+ 7.21	+ .03	− 1.16
100	.3	41	.9928	.0072	+208.17	−15.73	+15.36	+ 35.65	−26.50	− .44	− 4.28
100	.5	38	.0105	.0105	+351.61	+ 2.24	+32.34	− 36.35	−32.09	−1.47	− 7.11
100	.5	41	.0443	.0443	+153.40	+ .57	+19.03	− 34.25	+20.27	+ .16	− 6.02
100	.5	44	.1356	.1356	+ 62.98	+ .03	+ 9.48	− 30.99	+17.36	+ .46	− 4.48
100	.5	55	.8644	.1356	+ 58.91	+ .03	+11.66	+ 66.88	+20.05	+ .70	− 7.99
100	.5	58	.9557	.0443	+162.66	+ .57	+26.64	+115.29	+34.17	+ .37	−29.62
100	.5	61	.9895	.0105	+430.83	+ 2.24	+52.68	+197.46	− 1.21	−3.35	−86.70

proximations considered so far we have added the classical normal approximation

$$B_k \approx \Phi(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-\frac{1}{2}t^2}\,dt, \qquad \text{where} \quad u = \frac{k+\frac{1}{2}-\mu}{\sqrt{npq}}. \tag{3.8}$$

More numerical results, for more approximations, are given in tables 7–13.

4. Bomol and reversal rules

This section gives some results found by trial and error from numerical investigations. A new parameter δ is introduced, which is very accurate especially if we modify the rule of interchanging successes and failures if and only if* $p > .5$, and start this change (called *reversal*) for a somewhat smaller p if k is large, for a somewhat larger p if k is small.

Examination of BOLSHEV's approximation (3.5) in many numerical examples reveals that it is still conservative (with very rare exceptions for small p and n), though the overestimation of both tails is less severe than for the classical Poisson (3.1). The error increases with p as is illustrated in table 2.

In a numerical investigation the exact parameter value λ was computed for which $B_k = P_k(\lambda)$. As $P_k(\lambda)$ is a decreasing function of λ, the exact λ is unique and can be quickly found from successive linear interpolations**. It turned out that a closer approximation of λ is achieved if the last denominator $2-p$ in (3.7) is replaced by $2-3p/2$. The asymptotic behaviour for $p \to 0$ remains the same, but for larger p a somewhat stronger correction is imposed upon μ.

Thus the accuracy was investigated of

$$B_k \approx P_k(\delta) \quad \text{where} \quad \delta = \frac{(2-\frac{1}{2}p)\,n-k}{2-3p/2}\,p = \mu-(k-\mu)\frac{p}{2-3p/2}, \tag{4.1}$$

which was called Bomol as an abbreviation of BOLSHEV/MOLENAAR. In the examples considered it had a considerably smaller error than $P_k(\beta)$, which was already superior to $P_k(\mu)$. Even for $p = .5$ the relative error remains below 1 percent as long as $.01 < B_k < .70$ and $n \geqslant 40$. As the error in right hand tails is much larger than in left hand tails, a further improvement can be reached by the introduction of a new reversal rule:

$$B_k \approx \begin{cases} \displaystyle\sum_{j=0}^{k} \frac{\delta^j}{j!}e^{-\delta}, & \text{if} \begin{cases} p \leqslant .2+.016\sqrt{n} \text{ and } k \text{ arbitrary, or} \\ .2+.016\sqrt{n} < p < .8-.016\sqrt{n} \text{ and } k+1 \leqslant n/2; \end{cases} \\ \displaystyle\sum_{j=n-k}^{\infty} \frac{\eta^j}{j!}e^{-\eta} & \text{otherwise, where } \eta = \dfrac{(2-\frac{1}{2}q)\,n-n+k+1}{2-3q/2}\,q. \end{cases} \tag{4.2}$$

---

* "If and only if" is henceforth abbreviated by "iff".

** In the Algol 60 programme the procedure ZERO (AP 230) of the Mathematical Centre was used.

Table 3. Boundary values for reversal

n	$.2+.016\sqrt{n}$	$.8-.016\sqrt{n}$	n	$.2+.016\sqrt{n}$	$.8-.016\sqrt{n}$
4	.23	.77	70	.33	.67
10	.25	.75	100	.36	.64
20	.27	.73	140	.39	.61
30	.29	.71	200	.43	.57
50	.31	.69	300	.48	.52

The condition $p \leqslant .2+.016\sqrt{n}$ amounts to $p \leqslant .25$ for $n = 10$ and to $p \leqslant .40$ for $n = 160$, see also table 3. One could simplify (4.2) by replacing $.2+.016\sqrt{n}$ by .35 and $.8-.016\sqrt{n}$ by .65. This would lead to a few wrong decisions for, say, $.20 < p < .35$ and $.65 < p < .80$ if n is small, or for $.35 < p < .65$ if n is large.

Reversals have also been studied for the other Poisson approximations. For (3.1) and (3.2) it is difficult to give bounds for the region of (n, p, k) triplets for which reversal leads to a better approximation. For $P_k(\beta)$ reversal helps for $k < \frac{1}{2}n$ if $p=.5$, but even for $p = .45$ the advantage is dubious. Wise's $1-P_{n-k-1}(-\frac{1}{2}(n+k+1)\log p)$ is better than $P_k(\gamma)$ for roughly the values where (4.2) prescribes reversal. As in that region (4.2) is usually still better, no further attention has been given to the matter.

5. The exact parameter λ

The essential difference between (3.3) and (4.1) is that Bomol gives the cumulative Poisson distribution a better parameter instead of adding a correction term to (3.5). This is attractive as it simplifies calculations, but Bomol has a less spectacular numerical accuracy than (3.3). In order to find still better parameters, and to explain the success of Bomol, a theoretical counterpart is now derived for the numerical solution of the exact parameter λ from $B_k = P_k(\lambda)$.

We restrict ourselves to $0 \leqslant k < n$. It is well known that

$$B_k = \sum_{j=0}^{k} \binom{n}{j} p^j q^{n-j} = \int_p^1 \frac{n!}{k!\,(n-k-1)!} x^k (1-x)^{n-k-1} dx; \tag{5.1}$$

$$P_k(\lambda) = \sum_{j=0}^{k} \frac{\lambda^j}{j!} e^{-\lambda} = \int_\lambda^\infty \frac{1}{k!} x^k e^{-x} dx; \tag{5.2}$$

for a proof see e.g. FELLER (1957) chapter VI, problems 45 and 46. The exact λ depends on n, p and k; as a function of p it is infinitely differentiable for $0 < p < 1$ and even analytic in a neighbourhood of $p = 0$. We shall put $\lambda = \sum_{i=0}^{\infty} c_i p^i$, where the coefficients c_i depend on i, k and n but not on p. As for $p = 0$ and $\lambda = 0$ the integrals in (5.1) and (5.2) are equal to 1, it follows that $c_0 = 0$. Both integrals being equal for all p we may differentiate them with respect to p:

$$-\frac{n!}{k!\,(n-k-1)!}p^k(1-p)^{n-k-1} = -\frac{1}{k!}\lambda^k e^{-\lambda}\frac{d\lambda}{dp}, \tag{5.3}$$

which means

$$\frac{n!}{(n-k-1)!}(1-p)^{n-k-1} = \left(\frac{\lambda}{p}\right)^k e^{-\lambda}\frac{d\lambda}{dp}. \tag{5.4}$$

Both sides of (5.4) can be written as a power series in p, after substitution of $\lambda = \sum_{i=1}^{\infty} c_i p^i$. After some calculations this leads to

$$\begin{aligned}\frac{n!}{(n-k-1)!}\sum_{j=0}^{n-k-1}\binom{n-k-1}{j}(-p)^j &= c_1^{k+1}+\{(k+2)c_1^k c_2 - c_1^{k+2}\}\,p + \\ &+\{(k+3)c_1^k c_3+(k+3)\tfrac{1}{2}kc_1^{k-1}c_2^2-(k+3)c_1^{k+1}c_2+\tfrac{1}{2}c_1^{k+3}\}\,p^2 + \\ &+\{(k+4)c_1^k c_4+(k+4)kc_1^{k-1}c_2c_3-(k+4)c_1^{k+1}c_3+(k+4)\tfrac{1}{6}k(k-1)c_1^{k-2}c_2^3+ \\ &-(k+4)\tfrac{1}{2}(k+1)c_1^k c_2^2+(k+4)\tfrac{1}{2}c_1^{k+2}c_2-\tfrac{1}{6}c_1^{k+4}\}\,p^3+o(p^3) \qquad (p\to 0).\end{aligned} \tag{5.5}$$

As (5.5) is an identity in p, coefficients of the same power of p must be equal. In the notation $f_i = c_i/c_1$ $(i = 2,3,4)$ this implies

$$c_1 = \sqrt[k+1]{\frac{n!}{(n-k-1)!}}; \tag{5.6}$$

$$f_2 = \frac{c_1-n+k+1}{k+2}; \tag{5.7}$$

$$f_3 = \frac{(n-k-1)(n-k-2)-c_1^2}{2(k+3)} + c_2 - \tfrac{1}{2}kf_2^2; \tag{5.8}$$

$$\begin{aligned} f_4 = {}& \frac{c_1^3-(n-k-1)(n-k-2)(n-k-3)}{6(k+4)} + c_3 - kf_2f_3 + \\ & -\tfrac{1}{6}k(k-1)f_2^3+\tfrac{1}{2}(k+1)c_1f_2^2-\tfrac{1}{2}c_1c_2.\end{aligned} \tag{5.9}$$

We want to determine the asymptotic behaviour of the coefficients c_i for $n \to \infty$. As we want to exclude both degeneration in 0 and slipping away to infinity for our distribution, we make the assumption that the expectation μ has a limit between 0 and ∞. This means that also the region of interesting values of k, with essentially positive probability, is bounded. Our assumptions can be summarized as

$$n \to \infty,\ p \to 0,\ 0 < \lim \mu < \infty,\ k = O(1). \tag{5.10}$$

In equation (5.6) we may now apply STIRLING's formula

$$\log m! = (m+\tfrac{1}{2})\log m - m + \tfrac{1}{2}\log 2\pi + \frac{1}{12m} - \frac{1}{360m^3} + o(m^{-4}) \quad (m \to \infty), \tag{5.11}$$

and after a substantial calculation there results

$$c_1 = n - \frac{k}{2} - \frac{k^2+2k}{24n} - \frac{k^3+2k^2}{48n^2} - \frac{73k^4+172k^3+28k^2-48k}{5760n^3} + o(n^{-3}). \tag{5.12}$$

Combining this result with (5.7), (5.8) and (5.9) one obtains

$$c_2 = \frac{n}{2} - \frac{7k}{24} - \frac{k^2+2k}{48n} - \frac{21k^3+42k^2-8k}{1920n^2} + o(n^{-2}), \tag{5.13}$$

$$c_3 = \frac{n}{3} - \frac{5k}{24} - \frac{39k^2+76k}{2880n} + o(n^{-1}), \tag{5.14}$$

$$c_4 = \frac{n}{4} - \frac{469k}{2880} + o(1). \tag{5.15}$$

Similarly it follows

$$c_5 = \frac{n}{5} + o(n). \tag{5.16}$$

Now we collect terms which are of the same order under assumptions (5.10), such as p^2, p/n and $1/n^2$, and obtain a series expansion for the exact parameter λ displayed in the first line of table 4 (next page).

6. Expansions and new parameters

Now that we have obtained an expansion for λ, it can be compared to similar expansions for the approximating parameters β (3.5), γ (3.6) and δ (4.1). Table 4 shows that they all agree with λ in the order n^{-1}. We shall presently compare their second order differences, but let us first try to find some simple closed expression in n, p and k which, when similarly expanded, comes closer to λ. As both β and γ can be put in the form

$$\mu \cdot \left(\sum_{i=0}^{\infty} d_i p^i\right) \cdot \left(\sum_{j=0}^{\infty} e_j (k/n)^j\right) \tag{6.1}$$

we may try to improve them by finding coefficients d_i and e_j such that (6.1) is closer

Table 4. Series expansions for parameters

Exact	$\lambda = \mu\left\{1 + \frac{1}{2n}(\mu-k) + \frac{1}{24n^2}(8\mu^2-7\mu k-k^2-2k) + \frac{1}{288n^3}(72\mu^3-60\mu^2k-6\mu k^2-6k^3-12\mu k-12k^2) + \right.$
Mep	$\zeta = \mu\left\{1 + \frac{1}{2n}(\mu-k) + \frac{1}{24n^2}(8\mu^2-7\mu k-k^2-2k) + \frac{1}{288n^3}(64\mu^3-48\mu^2k-15\mu k^2-k^3-28\mu k+10k^2) + \right.$
Bomol	$\delta = \mu\left\{1 + \frac{1}{2n}(\mu-k) + \frac{1}{24n^2}(9\mu^2-9\mu k) + \frac{1}{288n^3}(81\mu^3-81\mu^2k) + \right.$
Bolshev	$\beta = \mu\left\{1 + \frac{1}{2n}(\mu-k) + \frac{1}{24n^2}(6\mu^2-6\mu k) + \frac{1}{288n^3}(36\mu^3-36\mu^2k) + \right.$
Wise	$\gamma = \mu\left\{1 + \frac{1}{2n}(\mu-k) + \frac{1}{24n^2}(8\mu^2-6\mu k) + \frac{1}{288n^3}(72\mu^3-48\mu^2k) + \right.$
Wimol	$\varepsilon = \mu\left\{1 + \frac{1}{2n}(\mu-k) + \frac{1}{24n^2}(8\mu^2-6\mu k-k^2) + \frac{1}{288n^3}(72\mu^3-48\mu^2k-6\mu k^2-6k^3) + \right.$

to λ than e.g. Wise's γ which has $d_i = 1/(i+1)$, $e_0 = 1$, $e_1 = -1/2$, $e_j = 0$ for $j \geqslant 2$. A better agreement is reached if we put $e_2 = -1/24$, $e_3 = -1/48$, keeping the other coefficients as they were. This means

$$B_k \approx \sum_{j=0}^{k} \frac{\varepsilon^j}{j!} e^{-\varepsilon} \qquad \text{where} \qquad \varepsilon = -\frac{24n^2-24nk+5k^2}{24n-12k} \log q. \tag{6.2}$$

The expansion of ε is found in table 4 under the name Wimol which stands for WISE/MOLENAAR. It gives a substantially better approximation than Wise, but is still unsatisfactory when k is large.

An expression of the form (6.1) can never agree with λ in the term of order n^{-2}. This agreement can be reached, however, by allowing p and k/n in both factors, and k/n^2 in one of them:

$$\zeta = \mu\cdot(1+d_1p+d_2k/n)\cdot(1+e_1p+e_2k/n+e_3p^2+e_4kp/n+e_5k^2/n^2+e_6k/n^2+\ldots). \tag{6.3}$$

Comparison with the expansion for λ in table 4 yields that complete agreement of second order terms is reached if we put $d_1 = -{}^1/_6$, $d_2 = -{}^7/_{12}$, $e_1 = {}^2/_3$, $e_2 = {}^1/_{12}$, $e_3 = {}^4/_9$, $e_4 = {}^1/_9$, $e_5 = {}^1/_{144}$, $e_6 = -{}^1/_{12}$. So finally we propose the approximation

$$B_k \approx \sum_{j=0}^{k} \frac{\zeta^j}{j!} e^{-\zeta} \qquad \text{where} \qquad \zeta = \frac{(12-2p)\,n-7k}{(12-8p)\,n-k+k/n}\, np, \tag{6.4}$$

which is called Mep (More Exact Parameter) in table 4. Its expansion coincides with the expansion of the exact parameter for order n^{-2} and even comes rather close to it in the third order term.

We shall use the series expansions in table 4 in order to draw conclusions about the accuracy of the five approximations β, γ, δ, ε and ζ of the exact Poisson para-

$$+\frac{1}{17280n^4}(3456\mu^4-2814\mu^3k-234\mu^2k^2-189\mu k^3-219k^4-456\mu^2k-378\mu k^2-516k^3+72\mu k-84k^2+144k)+\ldots\Big\}$$

$$+\frac{1}{17280n^4}(2560\mu^4-1600\mu^3k-840\mu^2k^2-115\mu k^3-5k^4-1600\mu^2k+680\mu k^2+110k^3+120k^2)\qquad+\ldots\Big\}\qquad(6.4)$$

$$+\frac{1}{17280n^4}(3645\mu^4-3645\mu^3k)\qquad+\ldots\Big\}\qquad(4.1)$$

$$-\frac{1}{17280n^4}(1080\mu^4-1080\mu^3k)\qquad+\ldots\Big\}\qquad(3.5)$$

$$+\frac{1}{17280n^4}(3456\mu^4-2160\mu^3k)\qquad+\ldots\Big\}\qquad(3.6)$$

$$+\frac{1}{17280n^4}(3456\mu^4-2160\mu^3k-240\mu^2k^2-180\mu k^3-180k^4)\qquad+\ldots\Big\}\qquad(6.2)$$

meter λ. Suppose we have found out for which values of n, p and k we have e.g. $|\delta-\lambda| < |\gamma-\lambda|$. This implies that $P_k(\delta)$ is closer to B_k than $P_k(\gamma)$ if $\delta-\lambda$ and $\gamma-\lambda$ have the same signs. In the case of opposite signs it does not strictly follow that $P_k(\delta)$ is a better approximation of B_k, but accuracy of a parameter will generally be a rather good indication of accuracy of its distribution function.

But how do we establish when $|\delta-\lambda| < |\gamma-\lambda|$ holds? We do have closed expressions for the approximating parameters, but not for λ. However, we have series expansions for all of them. If the difference in second order terms between δ and λ is smaller than the difference between γ and λ in the same order, this is almost equivalent to $|\delta-\lambda| < |\gamma-\lambda|$ provided that convergence is rapid enough to allow the neglection of terms of higher order.

Now many numerical examples, of which a few are given in table 5, indicate that convergence is indeed rapid even if n is not very large and p not very small. This table counts six triplets (n, p, k), and gives for each triplet the distribution function B_k and six sets of parameter values.

All six expansions in table 4 have the same leading term μ and the same first order value $\mu\{1-(k-\mu)/(2n)\}$; they are printed only once in table 5. On the next three lines are given, for λ, ζ, ε, δ, γ and β separately, the parameter values calculated from the expansions up to and including the second, third and fourth order. The last line marked "actual" gives in boldface type the actual values, towards which the expansions converge. These actual values are found for the exact parameter λ by an iterative process on the computer (see section 4), and for the approximating parameters from the closed expressions (6.4) for ζ, (6.2) for ε, (4.1) for δ, (3.6) for γ and (3.5) for β.

As an example we consider ε for $n = 40, p = .3$ and $k = 15$. The expansion values of order 0, 1, 2, 3, 4 are 12, 11.55, 11.5022, 11.4920 and 11.4897, respectively, while the actual value is 11.4890. From the second order values we would say that $\delta \approx 11.4488$ is closer to $\lambda \approx 11.4366$ than $\varepsilon \approx 11.5022$, while actually we have $\delta = 11.4194$, $\lambda = 11.3942$ and $\varepsilon = 11.4890$. Thus our conclusion "δ is better than ε"

Table 5. Some parameter values calculated from the series expansions of table 4, compared with the actual (limit) values

$n = 40$	$p = .3$	$k = 5$	$B_k = .0086$	$\mu = 12$	1st order = 13.0500	
	λ Exact	ζ Mep	ε Wimol	δ Bomol	γ Wise	β Bolshev
2nd order	13.2678	13.2678	13.2897	13.2863	13.2975	13.2075
3rd order	13.3184	13.3134	13.3465	13.3394	13.3560	13.2311
4th order	13.3307	13.3229	13.3606	13.3514	13.3704	13.2347
actual	**13.3350**	**13.3254**	**13.3654**	**13.3548**	**13.3753**	**13.2353**

$n = 40$	$p = .3$	$k = 8$	$B_k = .1110$	$\mu = 12$	1st order = 12.6000	
	λ Exact	ζ Mep	ε Wimol	δ Bomol	γ Wise	β Bolshev
2nd order	12.7250	12.7250	12.7600	12.7350	12.7800	12.6900
3rd order	12.7548	12.7518	12.8000	12.7654	12.8250	12.7035
4th order	12.7622	12.7576	12.8102	12.7722	12.8363	12.7055
actual	**12.7648**	**12.7592**	**12.8139**	**12.7742**	**12.8403**	**12.7059**

$n = 40$	$p = .3$	$k = 15$	$B_k = .8849$	$\mu = 12$	1st order = 11.5500	
	λ Exact	ζ Mep	ε Wimol	δ Bomol	γ Wise	β Bolshev
2nd order	11.4366	11.4366	11.5022	11.4488	11.5725	11.4825
3rd order	11.4063	11.4107	11.4920	11.4260	11.5860	11.4724
4th order	11.3978	11.4048	11.4897	11.4208	11.5903	11.4709
actual	**11.3942**	**11.4030**	**11.4890**	**11.4194**	**11.5919**	**11.4706**

$n = 40$	$p = .3$	$k = 18$	$B_k = .9852$	$\mu = 12$	1st order = 11.1000	
	γ Exact	ζ Mep	ε Wimol	δ Bomol	γ Wise	β Bolshev
2nd order	10.8750	10.8750	10.9538	10.8975	11.0550	10.9650
3rd order	10.8126	10.8224	10.9158	10.8519	11.0550	10.9448
4th order	10.7939	10.8101	10.9054	10.8417	11.0562	10.9417
actual	**10.7851**	**10.8064**	**10.9016**	**10.8387**	**11.0569**	**10.9412**

$n = 10$	$p = .5$	$k = 1$	$B_k = .0107$	$\mu = 5$	1st order = 6.0000	
	λ Exact	ζ Mep	ε Wimol	δ Bomol	γ Wise	β Bolshev
2nd order	6.3375	6.3375	6.3521	6.3750	6.3542	6.2500
3rd order	6.4658	6.4520	6.4869	6.5156	6.4896	6.3125
4th order	6.5176	6.4908	6.5414	6.5684	6.5443	6.3281
actual	**6.5559**	**6.5107**	**6.5819**	**6.6000**	**6.5849**	**6.3333**

$n = 10$	$p = .1$	$k = 3$	$B_k = .9872$	$\mu = 1$	1st order = .9000	
	λ Exact	ζ Mep	ε Wimol	δ Bomol	γ Wise	β Bolshev
2nd order	.8883	.8883	.8921	.8925	.8958	.8950
3rd order	.8867	.8875	.8911	.8919	.8956	.8947
4th order	.8864	.8875	.8909	.8919	.8956	.8947
actual	**.8864**	**.8875**	**.8909**	**.8919**	**.8956**	**.8947**

remains valid if we pass from second order parameter values to actual parameter values. As a next step we pass to the distribution functions and conclude that $P_k(\delta)$ is a better approximation of B_k than $P_k(\varepsilon)$ in this case. We shall see in the next section that the first few terms of the parameter expansions give nearly always correct conclusions about the accuracy of the corresponding Poisson approximations of binomial tails.

We shall now compare the second order terms in the expansions of table 4. We have

$$\begin{aligned}
(\beta-\lambda)/\mu &= b/(24n^2)+o(n^{-2}), \quad \text{where } b = k^2+k\mu-2\mu^2+2k;\\
(\gamma-\lambda)/\mu &= c/(24n^2)+o(n^{-2}), \quad \text{where } c = k^2+k\mu+2k;\\
(\delta-\lambda)/\mu &= d/(24n^2)+o(n^{-2}), \quad \text{where } d = k^2-2k\mu+\mu^2+2k;\\
(\varepsilon-\lambda)/\mu &= e/(24n^2)+o(n^{-2}), \quad \text{where } e = k\mu+2k;\\
(\zeta-\lambda)/\mu &= o(n^{-2}).
\end{aligned} \tag{6.5}$$

Evidently $c \geqslant 0, d > 0$ and $e \geqslant 0$, while $b > 0$ iff $k > k_1 = -1-\frac{1}{2}\mu+\sqrt{9\mu^2/4+\mu+1}$. The other root of $b = 0$ is negative and thus not interesting, as k assumes only non-negative integer values. One easily sees that $0 < k_1 < \mu$ holds.

Observe that $c < d$ iff $d-c = -3k\mu+\mu^2 > 0$, i.e. iff $k < \mu/3$. Similarly $d < e$ means $d-e = k^2-3k\mu+\mu^2 < 0$, i.e. $k_2 < k < k_3$ where $k_2 = 3\mu/2-\frac{1}{2}\mu\sqrt{5} \approx .38\mu$ and $k_3 = 3\mu/2+\frac{1}{2}\mu\sqrt{5} \approx 2.62\mu$. We always have $e < c$ unless $k = e = c = 0$. In that exceptional case we know that both Wise and Wimol give the exact probability q^n, while Mep has only zero in its second order error.

Neglecting a possible influence of higher order terms, we have now found the first four conclusions listed in table 6. The next two conclusions deal with Bolshev's difference b, and we must be careful because of its sign change. From $b-d = 3\mu(k-\mu)$ we see that for $k > \mu$ we have $b > d > 0$. On the other hand, $b+d = 2k^2+(4-\mu)k-\mu^2 = 0$ has a unique positive root $k_4 = -1+\frac{1}{4}\mu+\sqrt{9\mu^2/16-\frac{1}{2}\mu+1}$. As $d+b < 0$ implies $b < 0$ (because always $d > 0$), we have $|b| > d$ for $k < k_4$, and thus $|b| < d$ holds only in the strip $k_4 < k < \mu$.

Next observe that $b-c = -2\mu^2 < 0$, while $b+c = 2k^2+(4+2\mu)k-2\mu^2 = 0$ has a unique positive root $k_5 = -1-\frac{1}{2}\mu+\sqrt{5\mu^2/4+\mu+1}$. As $b+c < 0$ implies $b < 0$ we conclude that $|b| > c$ iff $k < k_5$. Finally $b-e = k^2-2\mu^2$ is positive for $k > \mu\sqrt{2}$ (so there $b > e > 0$) and $b+e = k^2+(2\mu+4)k-2\mu^2 = 0$ has a unique positive root $k_6 = -\mu+2+\sqrt{3\mu^2+4\mu+4}$, so we have $|b| < e$ only for $k_6 < k < k_7$ (we define $k_7 = \mu\sqrt{2}$).

Conclusion (vii) of table 6 follows from the first order difference:

$$(\mu-\lambda)/\mu = (k-\mu)/(2n)+o(n^{-1}). \tag{6.6}$$

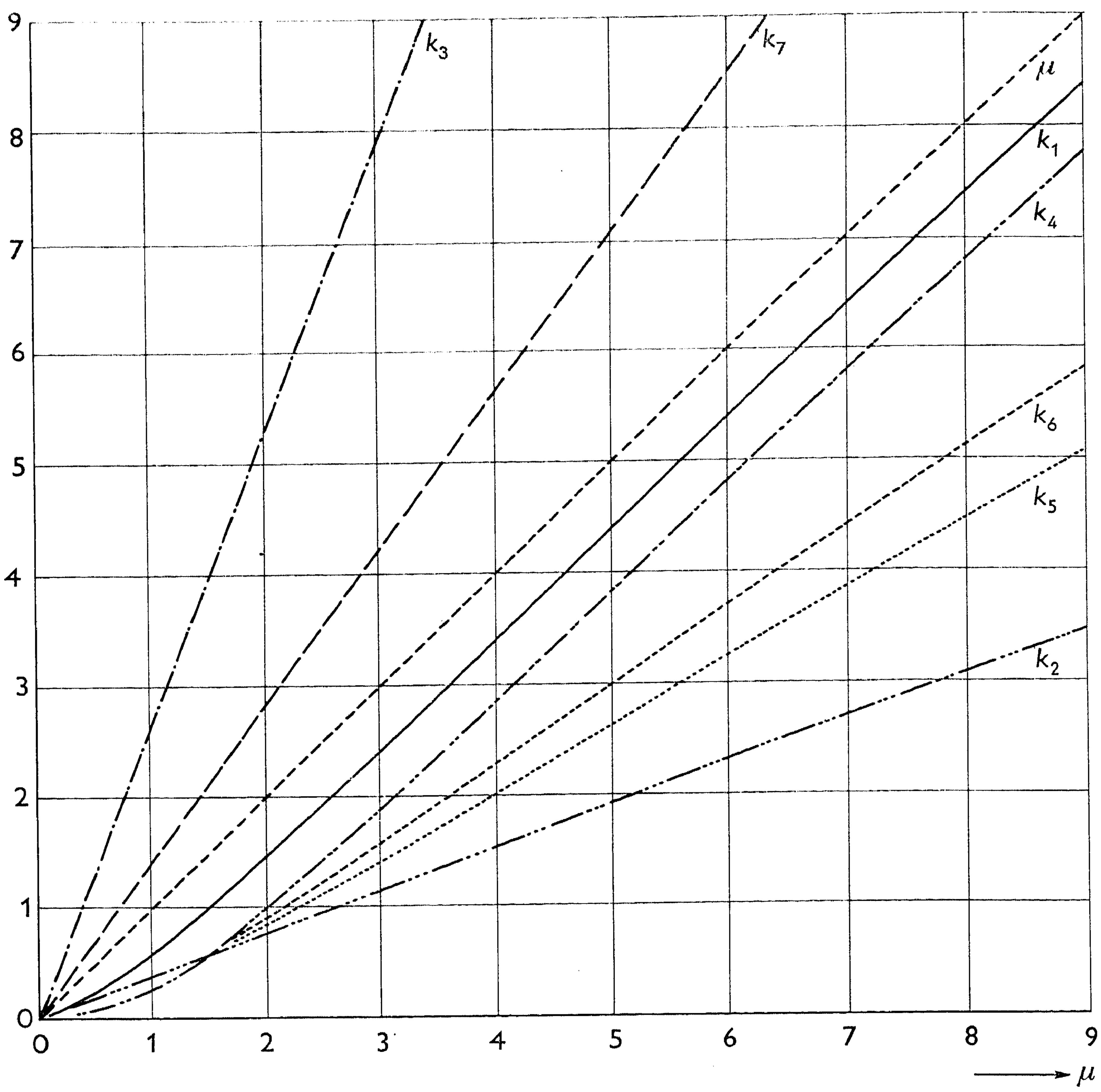

Fig. 1. Values of k_i, mentioned in table 6, as functions of the expectation μ

Table 6. Conclusions from the differences in parameter expansions. The symbols k_i are defined in the text and their values as functions of μ are sketched in fig. 1.

(i)	Mep is superior to any of the other approximations, except for $k = 0$ where Wise and Wimol are preferable.
(ii)	Bolshev's parameter is too small for $k < k_1$ and too large for $k > k_1$, thus overestimating both tails. The parameters of Wise, Bomol and Wimol are always too large, which means underestimation of the left hand tail and overestimation of the right hand tail.
(iii)	Wise is only superior to Bomol for $k < \mu/3$, and Wise is never superior to Wimol.
(iv)	Bomol is superior to Wimol in a middle region $k_2 < k < k_3$ containing most of the distribution
(v)	Bomol is superior to Bolshev except in a narrow strip $k_4 < k < \mu$.
(vi)	Wise is superior to Bolshev iff k $< k_5$ (in the far left hand tail). Wimol is superior to Bolshev outside of a rather wide region $k_6 < k < k_7$.
(vii)	The classical Poisson is conservative, it overestimates B_k for $k < \mu$ and underestimates it for $k > \mu$.
(viii)	Mep is nearly always conservative, as it overestimates B_k for $k \leqslant \max(\mu - 1{,}0)$ and underestimates it for $k \geqslant \mu$.

The conservatism of Mep stated in conclusion (viii) is deduced from the third-order difference

$$(\zeta-\lambda)/\mu = z/(288n^3)+o(n^{-3}), \tag{6.7}$$

where

$$z = -8\mu^3+12\mu^2k-9\mu k^2+5k^3-16\mu k+22k^2 =$$
$$= (k-\mu)\,(5k^2-4k\mu+8\mu^2+16k)+6k^2. \tag{6.8}$$

As the second term is nonnegative and the first term has the sign of $k-\mu$, it follows that $z > 0$ for $k \geqslant \mu$. For $k = 0$ we have $z = -8\mu^3 < 0$, while in general we have

$$z = (k-\mu-1)\,(8\mu^2-4k\mu+5k^2)-8\mu^2-12\mu k+17k^2. \tag{6.9}$$

For $1 \leqslant k < \mu-1$ the first product consists of a negative and a positive factor, while the remaining terms have a negative sum. Thus $z < 0$ for $k \leqslant \max(0,\mu-1)$. A numerical investigation shows that the sign change of z occurs for $\mu = 1.21$ if $k = 1$, for $\mu = 2.31$ if $k = 2$, for $\mu = 5.44$ if $k = 5$, for $\mu = 10.53$ if $k = 10$, and between $\mu = k+.6$ and $\mu = k+.7$ if $k \geqslant 25$.

7. Numerical results

The conclusions of table 6 have been deduced under neglection of terms of higher order. A numerical investigation exhibits that they are rarely wrong in finite situations. For $n = 5$, 10, 20, 30, 50 100, 300 and $p = .01$, .03, .05 (.05) .50 all values of k were considered for which $.001 < B_k < .999$. This means 7 times 12 = 84 parameter pairs (n,p) and some 1300 values for k. We shall indicate for each conclusion when it was found to be wrong.

Ad (i): Mep was never inferior to Poisson, only for two triplets (n, p, k) inferior to Bolshev, only for three triplets with $k > 0$ inferior to Wise, only for six triplets with $k > 0$ inferior to Wimol. However for $p = .5$ and $n = 10$, for $p \geqslant .4$ and $n = 30$, for $p \geqslant .3$ and $n \geqslant 100$, and similar p for other n, a region (say about $.001 < B_k < .35$) existed where Bomol is better than Mep. The difference in accuracy is negligible in most of this region, but Bomol is substantially more accurate for small B_k, large n and p near .5.

Ad (ii): In 4 of the 84 distributions just one value of k slightly larger than k_1 existed for which Bolshev's parameter was still too small; but for small tails Bolshev was always conservative. The Bomol parameter was sometimes too small for seven distributions, namely $n = 50$, $p = .5$, $.03 < B_k < .18$; $n = 300$, $p \geqslant .35$, $.001 < B_k < .35$, and intermediate regions for $n = 100$. No violations of the other statements in (ii) have been observed.

Ad (iii): Only twice a value $k < \mu/3$ was found for which Bomol was superior to Wise, and the difference in accuracy was small. Wise was indeed never seen to be superior to Wimol.

Ad (iv): For $n = 20$ and 30, and p near .5, six distributions were found for which the boundary k_2 was a little bit too large. For twelve distributions, mainly for small n and p near .5, the boundary k_3 was a little bit too large. This led on each occasion to an unjustified preference for Wimol over Bomol for just one value of k, which is anyhow not very relevant as Mep is in these cases more accurate than both of them.

Ad (v): In one quarter of the distributions, mainly with p near .5, the narrow strip $k_4 < k < \mu$ was not narrow enough. The statement "Bomol is always better than Bolshev" thus admits still less exceptions than would follow from (v).

Ad (vi): Again one quarter of the distributions, mainly for p near .5 and small n, showed a too small k_5: thus values $k > k_5$ existed for which Wise was still better than Bolshev. In almost half of the distributions (again for the larger p) the wide strip $k_6 < k < k_7$ was somewhat too wide, i.e. for some k Wimol was better than Bolshev while it was not predicted to be.

Ad (vii): For $n \leqslant 50$ most distributions exhibited one value for k where the Poisson approximation was not conservative. But this was just an internal contradiction in rule (vii), as it always was a value $k < \mu$ for which nevertheless $B_k > .5$. Nearly all these cases had $.5 < B_k < .6$ and conservatism in the small tails was maintained. For very small n and p it may happen that some $B_k > .9$ is overestimated for a $k < \mu$, but there the error will be small as p is small.

Ad (viii): Only for seven distributions, with small n, just one value of k existed for which Mep was not conservative but had a tail error just below zero. This is no violation of rule (viii) as it always happened for $\max(0, \mu - 1) < k < \mu$.

Summarizing these findings we may say that the conclusions of table 6 give a fairly good picture of the accuracy of the various approximations by Poisson distribution functions, though they are based on expansions of their parameters with terms of higher order neglected.

Tables 7–13 give the relative tail accuracy (2.2) for a set of triplets (n, p, k) which were selected to give a good impression of what happens for the many other triplets for which accuracies were calculated by the computer.

Each line of the tables corresponds to one value of k in the distribution with the indicated parameters n and p. Whenever $B_k \leqslant .5$ the line gives a $\leqslant$ sign, k, $B_k = P\{\underline{x} \leqslant k\}$, and $100(A_k - B_k)/B_k$ for eight approximations A_k of which the formula number in this paper is indicated at the top of each table. For $B_k > .5$ the line gives a $\geqslant$ sign, $k+1$, $1 - B_k = P\{\underline{x} \geqslant k+1\}$ and $100[(1-A_k)-(1-B_k)]/(1-B_k)$ for the same eight A_k. The relative error of the reversed Bomol (4.2) is not printed when it coincides with the ordinary Bomol.

Consider for example the binomial distribution with $n = 100$ and $p = .1$ in table 10. The second line tells us that $P\{\underline{x} \leqslant 4\} = .0237$ is overestimated by the normal approximation with a relative error of 40.76 percent, so this approximation would give the value $(140.76)\cdot(.0237) = .0334$. The Poisson approximation with parameter

np overestimates B_4 by 23.37 percent, so it gives $(123.37)\cdot(.0237) = .0292$. Its correction by the first Gram-Charlier term leads to an underestimation by .56 percent, Bolshev overestimates by .43 percent and the other approximations are still considerably better.

In table 12 one can see that even for $n = 100$ and $p = .5$ the classical normal approximation is less accurate than Bomol with reversal. The Gram-Charlier correction is not very adequate for p near .5, but Bolshev and especially Bomol and Mep present a spectacular gain in accuracy.

Bomol and Mep are both rather accurate. Bomol is slightly easier to compute and is better near $p = .5$, where in general Poisson-type approximations have low accuracy. Mep has the advantage of being a little bit conservative, and it does not need a complicated reversal rule for right hand tails. From our study Mep emerges as a very accurate and relatively simple Poisson-type approximation.. Even if a bit of the enormous gain in accuracy is lost again when roughly interpolating in a Poisson table, it seems very desirable to use Mep in stead of the classical Poisson approximation.

Acknowledgement

Many thanks are due to Professor VAN ZWET, for frequent discussions and valuable suggestions. Professor HEMELRIJK kindly stimulated this investigation and allowed the use of the Electrologica X8 computer of the Mathematical Centre. Mr. VEHMEYER checked the computations leading to table 4.

References

BOLSHEV, L. N., B. V. GLADKOV and M. V. SHCHEGLOVA, Tables for the calculation of B- and z-distribution functions. Theory of Prob. and its Appl. **6** (1961), 410–419 (English translation from Teoriya Veroyatnostei i ee Primeneniya).

BORGES, R., Eine Approximation der Binomialverteilung durch die Normalverteilung der Ordnung 1/n. Submitted to Zeitschr. für Wahrsch. Theorie und verwandte Gebiete.

FELLER, W., An Introduction to Probability Theory and its Applications. Vol. 1, 2nd edition, Wiley, New York 1957.

FREEMAN, M. F. and J. W. TUKEY, Transformations related to the angular and the square root. Ann. Math. Stat. **21** (1950), 607–611.

GENERAL ELECTRIC COMPANY, Tables of the Individual and Cumulative Terms of Poisson Distribution. Van Nostrand, Princeton (N.J.) 1962.

HALD, A., Statistical Theory with Engineering Applications. Wiley, New York 1952.

HODGES Jr., J. L. and L. LECAM, The Poisson approximation to the Poisson binomial distribution, Ann. Math. Stat. **31** (1960), 737–740.

LECAM, L., An approximation theorem for the Poisson binomial distribution. Pacific J. of Math. **10** (1960), 1181–1197.

MOLINA, E. C., Poisson's Exponential Binomial Limit, Van Nostrand, New York 1945.

ORDNANCE CORPS, Tables of the Cumulative Binomial Probabilities. Document ORDP 20-1 of the Ordnance Corps of the U.S. Army (1952).

PEARSON, E. S. and H. O. HARTLEY, Biometrika Tables for Statisticians. Vol. 1, Cambridge Univ. Press, 1954.

PROHOROV, YU. V., Asymptotic behavior of the binomial distribution. Uspehi Mat. Nauk (N.S.) **8** (1953) no 3 (35), 135–142 (Russian); English translation in Selected Translations in Math. Stat. and Probability, vol. 1, Amer. Math. Soc., Providence (R.I.) 1961, 87–96.

RAFF, M. S., On approximating the point binomial. J. of the Amer. Stat. Assoc. **51** (1956), 293–303.

VERENIGING VOOR STATISTIEK, Statistische Tabellen en Nomogrammen. Stenfert Kroese, Leiden.

WEINTRAUB, S., Tables of the Cumulative Binomial Probability Distribution for Small Values of p. The Free Press of Glencoe, New York 1963.

WISE, M. E., The use of the negative binomial distribution in an industrial sampling problem. J. of the Royal Stat. Soc. Supplement **8** (1946), 202–211.

WISE, M. E., The incomplete Beta function as a contour integral and a quickly converging series for its inverse. Biometrika **37** (1950), 208–218.

Table 7. Relative errors in percentages of the binomial tail, cf. (2.2)

Region	Tail	Normal (3.8)	Poisson (3.1)	GR.CH (3.2)	Bolshev (3.5)	Wise (3.6)	Wimol (6.2)	Mep (6.4)	Bomol (4.1)	RVBML (4.2)
$N = 10$	$P = .01$									
≥ 1	.0956	+ 6.48	− .48	− .00	− .00	+ .00	+ .00	− .00	+ .00	
≥ 2	.0043	−99.90	+ 9.67	+ .13	+ .28	+ .28	+ .19	+ .02	+ .25	
$N = 10$	$P = .05$									
≥ 1	.4013	+24.61	− 1.94	− .05	− .02	− .00	− .00	− .00	+ .01	
≥ 2	.0861	−14.79	+ 4.72	+ .32	+ .24	+ .27	+ .19	+ .01	+ .18	
≥ 3	.0115	−83.88	+25.07	+ .36	+ 1.18	+ 1.24	+ .68	+ .12	+ .89	
≥ 4	.0010	−99.34	+70.31	− 6.48	+ 3.47	+ 3.55	+ 1.60	+ .53	+ 2.75	
$N = 10$	$P = .10$									
≤ 0	.3487	−14.22	+ 5.51	+ .23	+ .10	− .00	− .00	+ .00	− .04	
≥ 2	.2639	+13.33	+ .13	+ .13	+ .13	+ .26	+ .19	+ .00	+ .13	
≥ 3	.0702	−18.90	+14.40	+ 1.30	+ .98	+ 1.20	+ .71	+ .08	+ .63	
≥ 4	.0128	−67.14	+48.40	+ .48	+ 3.17	+ 3.48	+ 1.72	+ .42	+ 2.08	
≥ 5	.0016	−93.12	+123.85	−16.78	+ 7.79	+ 8.22	+ 3.44	+ 1.39	+ 5.45	
$N = 10$	$P = .20$									
≤ 0	.1074	+ 9.75	+26.04	+ .83	+ .93	− .00	− .00	+ .07	− .39	
≤ 1	.3758	− 7.85	+ 8.03	+ .83	+ .25	− .34	− .27	+ .03	− .19	
≥ 3	.3222	+ 7.48	+ .35	+ .35	+ .35	+ 1.05	+ .70	+ .02	+ .35	
≥ 4	.1209	− 2.51	+18.20	+ 3.27	+ 2.09	+ 3.20	+ 1.81	+ .24	+ 1.18	
≥ 5	.0328	−26.65	+60.56	+ 5.53	+ 6.18	+ 7.77	+ 3.79	+ .95	+ 3.40	
≥ 6	.0064	−55.59	+160.05	− 9.93	+14.64	+16.85	+ 7.17	+ 2.86	+ 8.53	
$N = 10$	$P = .30$									
≤ 0	.0282	+49.57	+76.25	− 3.06	+ 3.80	− .00	− .00	+ .43	− 1.38	
≤ 1	.1493	+ .67	+33.38	+ 3.37	+ 1.98	− .77	− .65	+ .23	− .67	
≤ 2	.3828	− 4.64	+10.56	+ 1.78	+ .54	− 1.30	− .94	+ .08	− .40	
≥ 4	.3504	+ 4.18	+ .68	+ .68	+ .68	+ 2.71	+ 1.70	+ .06	+ .68	
≥ 5	.1503	+ .03	+22.94	+ 6.16	+ 3.82	+ 6.96	+ 3.83	+ .56	+ 2.04	
≥ 6	.0473	−10.77	+77.23	+13.35	+11.03	+15.63	+ 7.58	+ 2.04	+ 5.58	+ 1.93
≥ 7	.0106	−25.77	+216.35	+ 2.19	+26.37	+33.08	+14.14	+ 5.89	+13.92	+ 1.41
≥ 8	.0016	−40.24	+648.53	−166.52	+60.29	+70.76	+26.55	+15.60	+33.50	− .98

Table 8. Relative errors in percentages of the binomial tail, cf. (2.2)

Region	Tail	Normal (3.8)	Poisson (3.1)	GR.CH (3.2)	Bolshev (3.5)	Wise (3.6)	Wimol (6.2)	Mep (6.4)	Bomol (4.1)	RVBML (4.2)
$N = 10$	$P = .40$									
≤ 0	.0060	+97.37	+202.91	−39.42	+11.43	+ .00	+ .00	+ 1.75	− 3.40	
≤ 1	.0464	+14.96	+97.55	+ 2.73	+ 7.31	− 1.44	− 1.26	+ 1.10	− 1.79	
≤ 2	.1673	− .50	+42.33	+ 7.29	+ 3.76	− 2.62	− 2.01	+ .60	− .95	
≤ 3	.3823	− 2.31	+13.39	+ 3.17	+ 1.03	− 3.30	− 2.25	+ .20	− .67	
≥ 5	.3669	+ 1.78	+ 1.16	+ 1.16	+ 1.16	+ 5.77	+ 3.51	+ .15	+ 1.16	
≥ 6	.1662	+ .13	+29.25	+10.45	+ 6.54	+13.74	+ 7.44	+ 1.23	+ 3.44	− .83
≥ 7	.0548	− 2.68	+102.10	+25.99	+19.22	+30.18	+14.54	+ 4.32	+ 9.42	−1.29
≥ 8	.0123	− 2.93	+315.90	+25.34	+48.53	+66.09	+28.04	+12.52	+24.41	−2.97
≥ 9	.0017	+ 9.54	+1173.36	−246.19	+126.68	+159.78	+58.34	+35.83	+65.47	−6.01
$N = 10$	$P = .45$									
≤ 0	.0025	+117.22	+338.58	−105.48	+18.76	− .00	− .00	+ 3.25	− 4.87	
≤ 1	.0233	+21.53	+162.71	− 6.56	+12.67	− 1.91	− 1.69	+ 2.15	− 2.68	
≤ 2	.0996	+ 2.26	+74.35	+10.80	+ 7.35	− 3.53	− 2.78	+ 1.29	− 1.40	
≤ 3	.2660	− 1.33	+28.66	+ 7.26	+ 3.07	− 4.62	− 3.26	+ .60	− .88	
≥ 5	.4956	+ .89	− 5.59	− 1.28	− .08	+ 5.04	+ 3.20	− .06	+ .85	
≥ 6	.2616	+ .36	+13.57	+ 6.23	+ 4.18	+12.52	+ 7.13	+ .80	+ 2.60	−1.01
≥ 7	.1020	− .18	+65.64	+23.25	+15.21	+28.25	+14.39	+ 3.50	+ 7.55	−1.31
≥ 8	.0274	+ 3.19	+216.10	+46.97	+41.69	+62.94	+28.34	+10.97	+20.50	−2.55
≥ 9	.0045	+22.21	+794.16	−16.20	+113.33	+153.77	+59.70	+32.61	+56.78	−4.94
$N = 10$	$P = .50$									
≤ 0	.0010	+126.63	+589.97	−272.49	+30.32	− .00	− .00	+ 5.81	− 6.62	
≤ 1	.0107	+25.01	+276.34	−37.28	+21.25	− 2.48	− 2.23	+ 3.99	− 3.76	
≤ 2	.0547	+ 4.09	+127.94	+12.43	+13.31	− 4.65	− 3.76	+ 2.55	− 1.96	
≤ 3	.1719	− .28	+54.20	+13.36	+ 6.77	− 6.25	− 4.57	+ 1.39	− 1.11	
≤ 4	.3770	− .28	+16.86	+ 5.22	+ 1.89	− 7.01	− 4.63	+ .46	− .97	
≥ 6	.3770	− .28	+ 1.88	+ 1.88	+ 1.88	+11.11	+ 6.64	+ .34	+ 1.88	− .97
≥ 7	.1719	− .28	+38.37	+17.10	+11.06	+25.97	+13.96	+ 2.64	+ 5.87	−1.11
≥ 8	.0547	+ 4.09	+143.88	+48.39	+34.39	+59.15	+28.25	+ 9.36	+16.92	−1.96
≥ 9	.0107	+25.01	+533.89	+78.13	+98.87	+146.49	+60.54	+29.33	+48.80	−3.76

Table 9. Relative errors in percentages of the binomial tail, cf. (2.2)

Region	Tail	Normal (3.8)	Poisson (3.1)	Gr.CH (3.2)	Bolshev (3.5)	Wise (3.6)	Wimol (6.2)	Mep (6.4)	Bomol (4.1)	RVBML (4.2)
$N = 40$	$P = .01$									
≥ 1	.3310	+31.97	− .41	− .00	− .00	+ .00	+ .00	− .00	+ .00	
≥ 2	.0607	−33.76	+ 1.34	+ .02	+ .01	+ .02	+ .01	+ .00	+ .01	
≥ 3	.0075	−94.35	+ 5.72	− .00	+ .06	+ .07	+ .04	+ .00	+ .05	
$N = 40$	$P = .10$									
≤ 0	.0148	+120.17	+23.91	− .87	+ .39	− .00	− .00	+ .01	− .18	
≤ 1	.0805	+16.58	+13.80	+ .14	+ .25	− .06	− .06	+ .01	− .10	
≤ 2	.2228	− 3.68	+ 6.86	+ .29	+ .12	− .11	− .09	+ .00	− .05	
≤ 3	.4231	− 6.39	+ 2.44	+ .13	+ .03	− .14	− .09	+ .00	− .04	
≥ 5	.3710	+ 6.76	+ .05	+ .05	+ .05	+ .24	+ .15	+ .00	+ .05	
≥ 6	.2063	+ 4.04	+ 4.17	+ .38	+ .21	+ .48	+ .26	+ .01	+ .10	
≥ 7	.0995	− 5.73	+11.21	+ .74	+ .48	+ .84	+ .42	+ .02	+ .20	
≥ 8	.0419	−22.33	+22.03	+ .72	+ .91	+ 1.36	+ .62	+ .05	+ .38	
≥ 9	.0155	−42.87	+37.87	− .55	+ 1.53	+ 2.07	+ .87	+ .09	+ .66	
≥ 10	.0051	−63.00	+60.62	− 4.71	+ 2.37	+ 3.01	+ 1.17	+ .16	+ 1.07	
≥ 11	.0015	−79.14	+93.22	−14.81	+ 3.50	+ 4.24	+ 1.53	+ .27	+ 1.66	
$N = 40$	$P = .30$									
≤ 4	.0026	+88.61	+196.79	−51.97	+ 8.44	− 2.20	− 1.76	+ .79	− 1.83	
≤ 6	.0238	+21.50	+92.85	− 3.67	+ 5.25	− 2.97	− 2.14	+ .49	− .90	
≤ 8	.1110	+ 2.33	+39.65	+ 4.24	+ 2.68	− 3.35	− 2.19	+ .25	− .42	
≤ 10	.3087	− 2.06	+12.47	+ 2.28	+ .84	− 3.27	− 1.95	+ .09	− .24	
≥ 13	.4228	+ 2.06	+ .29	+ .29	+ .29	+ 3.72	+ 2.02	+ .03	+ .29	
≥ 15	.1926	+ .85	+18.40	+ 4.30	+ 2.32	+ 7.81	+ 3.87	+ .25	+ .82	
≥ 17	.0633	− 4.83	+59.98	+ 8.53	+ 6.53	+14.59	+ 6.56	+ .78	+ 2.09	
≥ 19	.0148	−15.70	+153.21	− 2.40	+14.02	+25.38	+10.33	+ 1.83	+ 4.65	
≥ 21	.0024	−30.58	+379.37	−100.86	+26.63	+42.45	+15.49	+ 3.74	+ 9.35	
$N = 40$	$P = .50$									
≤ 10	.0011	+19.88	+873.40	−435.73	+46.01	−16.65	−12.34	+ 6.88	− 2.32	
≤ 12	.0083	+ 6.73	+370.34	−54.65	+30.42	−18.10	−12.68	+ 4.75	− .82	
≤ 14	.0403	+ 1.61	+159.92	+15.90	+18.21	−18.61	−12.30	+ 2.99	− .14	
≤ 16	.1341	+ .07	+64.87	+16.72	+ 9.12	−18.00	−11.20	+ 1.58	− .08	
≤ 18	.3179	− .09	+19.98	+ 6.70	+ 2.91	−16.16	− 9.41	+ .54	− .34	
≥ 22	.3179	− .09	+12.08	+ 5.42	+ 3.29	+24.11	+12.50	+ .62	+ 1.55	− .34
≥ 24	.1341	+ .07	+58.48	+21.07	+12.59	+46.47	+21.91	+ 2.52	+ 4.44	− .08
≥ 26	.0403	+ 1.61	+178.06	+39.92	+31.40	+86.43	+36.24	+ 6.41	+10.57	− .14
≥ 28	.0083	+ 6.73	+532.72	− 3.31	+68.44	+161.92	+58.40	+13.86	+22.71	− .82
≥ 30	.0011	+19.88	+1864.34	−670.91	+145.53	+319.84	+94.43	+27.99	+46.69	− 2.32

Table 10. Relative errors in percentages of the binomial tail, cf. (2.2)

Region	Tail	Normal (3.8)	Poisson (3.1)	GR.CH (3.2)	Bolshev (3.5)	Wise (3.6)	Wimol (6.2)	Mep (6.4)	Bomol (4.1)	RVBML (4.2)
$N = 100$	$P = .01$									
≤ 0	.3660	−15.95	+ .50	+ .00	+ .00	− .00	− .00	+ .00	− .00	
≥ 2	.2642	+16.43	+ .00	+ .00	+ .00	+ .00	+ .00	+ .00	+ .00	
≥ 3	.0794	−17.06	+ 1.17	+ .01	+ .01	+ .01	+ .01	+ .00	+ .00	
≥ 4	.0184	−67.39	+ 3.34	+ .01	+ .02	+ .02	+ .01	+ .00	+ .01	
≥ 5	.0034	−93.65	+ 6.63	− .07	+ .05	+ .05	+ .02	+ .00	+ .03	
$N = 100$	$P = .05$									
≤ 0	.0059	+228.92	+13.81	− .42	+ .11	− .00	− .00	+ .00	− .05	
≤ 2	.1183	+ 6.27	+ 5.40	+ .06	+ .05	− .03	− .02	+ .00	− .02	
≤ 4	.4360	− 6.13	+ 1.03	+ .03	+ .01	− .04	− .02	+ .00	− .01	
≥ 7	.2340	+ 4.98	+ 1.64	+ .07	+. 04	+ .11	+ .06	+ .00	+ .02	
≥ 9	.0631	−14.17	+ 7.93	+ .17	+ .16	+ .26	+ .12	+ .00	+ .06	
≥ 11	.0115	−49.37	+19.38	− .38	+ .38	+ .53	+ .21	+ .01	+ .15	
≥ 13	.0015	−80.22	+37.87	−3.17	+ .74	+ .93	+ .34	+ .03	+ .32	
$N = 100$	$P = .10$									
≤ 2	.0019	+219.28	+42.39	−4.29	+ .69	− .11	− .09	+ .02	− .25	
≤ 4	.0237	+40.76	+23.37	− .56	+ .43	− .19	− .14	+ .01	− .13	
≤ 6	.1172	+ 3.86	+11.08	+ .32	+ .22	− .24	− .16	+ .01	− .06	
≤ 8	.3209	− 3.84	+ 3.72	+ .21	+ .07	− .25	− .15	+ .00	−. 03	
≥ 12	.2970	+ 3.90	+ 2.11	+ .19	+ .10	+ .45	+ .23	+ .00	+ .04	
≥ 14	.1239	− 1.78	+ 9.41	+ .58	+ .34	+ .85	+ .41	+. 01	+ .10	
≥ 16	.0399	−16.33	+22.19	+ .43	+ .76	+1.45	+ .63	+ .03	+ .22	
≥ 18	.0100	−37.95	+42.67	−1.97	+1.40	+2.27	+ .92	+ .06	+ .44	
≥ 20	.0020	−61.03	+74.59	−10.29	+2.31	+3.38	+1.27	+ .11	+ .78	
$N = 100$	$P = .20$									
≤ 9	.0023	+85.65	+114.07	−23.02	+3.67	−1.70	−1.23	+ .21	− .74	
≤ 11	.0126	+33.55	+70.08	−5.61	+2.62	−1.89	−1.28	+ .15	− .45	
≤ 13	.0469	+11.02	+40.96	+ .50	+1.72	−1.97	−1.26	+ .10	− .26	
≤ 15	.1285	+ 1.39	+21.79	+1.70	+ .99	−1.94	−1.17	+ .06	− .15	
≤ 17	.2712	− 1.92	+ 9.53	+1.13	+ .43	−1.80	−1.02	+ .03	− .09	
≤ 19	.4602	− 2.15	+ 2.19	+ .26	+ .06	−1.54	− .83	+ .00	− .07	
≥ 23	.2611	+ 1.88	+ 7.02	+1.13	+ .55	+2.92	+1.45	+ .03	+ .17	
≥ 25	.1314	− .81	+19.35	+2.38	+1.34	+4.50	+2.12	+ .09	+ .34	
≥ 27	.0558	− 6.72	+39.50	+2.64	+2.55	+6.58	+2.95	+ .18	+ .62	
≥ 29	.0200	−16.12	+71.49	−1.02	+4.27	+9.25	+3.93	+ .31	+1.08	
≥ 31	.0061	−28.50	+122.38	−15.32	+6.58	+12.61	+5.07	+ .51	+1.75	
≥ 33	.0016	−42.66	+204.91	−55.48	+9.61	+16.82	+6.40	+ .79	+2.71	

Table 11. Relative errors in percentages of the binomial tail, cf. (2.2)

Region	Tail	Normal (3.8)	Poisson (3.1)	GR.CH (3.2)	Bolshev (3.5)	Wise (3.6)	Wimol (6.2)	Mep (6.4)	Bomol (4.1)	RVBML (4.2)
$N = 100$	$P = .30$									
≤ 17	.0022	+47.42	+236.13	−70.17	+10.53	− 6.95	− 4.71	+ .89	−1.15	
≤ 20	.0165	+15.92	+114.33	− 7.87	+ 6.68	− 7.14	− 4.55	+ .57	− .56	
≤ 23	.0755	+ 3.31	+ 51.79	+ 4.41	+ 3.65	− 6.84	− 4.11	+ .32	− .25	
≤ 26	.2244	− .84	+ 19.13	+ 3.36	+ 1.47	− 6.05	− 3.42	+ .13	− .15	
≤ 29	.4623	− 1.25	+ 2.89	+ .54	+ .13	− 4.82	− 2.56	+ .01	− .14	
≥ 33	.2893	+ 1.18	+ 9.05	+ 2.22	+ 1.11	+ 8.15	+ 4.06	+ .11	+ .35	
≥ 36	.1161	− .90	+ 35.58	+ 6.31	+ 3.71	+14.25	+ 6.64	+ .38	+ .90	
≥ 39	.0340	− 6.39	+ 90.84	+ 5.48	+ 8.23	+23.09	+10.00	+ .87	+1.99	
≥ 42	.0072	−15.73	+208.17	−26.50	+15.36	+35.65	+14.26	+1.69	+3.89	
≥ 45	.0011	−28.38	+477.36	−193.13	+26.11	+53.47	+19.58	+2.96	+6.95	
$N = 100$	$P = .40$									
≤ 25	.0012	+29.44	+536.36	−241.70	+26.29	−18.25	−12.28	+2.96	−1.12	
≤ 28	.0084	+12.09	+248.41	−37.36	+17.46	−18.17	−11.68	+2.04	− .41	
≤ 31	.0398	+ 3.81	+114.62	+ 6.99	+10.50	−17.35	−10.64	+1.27	− .09	
≤ 34	.1303	+ .34	+ 48.75	+ 9.65	+ 5.30	−15.73	− 9.18	+ .67	− .04	
≤ 37	.3068	− .62	+ 15.59	+ 4.19	+ 1.76	−13.32	− 7.37	+ .23	− .14	
≥ 41	.4567	+ .58	+ .30	+ .30	+ .30	+12.20	+ 6.37	+ .04	+ .30	
≥ 44	.2365	+ .40	+ 19.97	+ 6.17	+ 3.24	+22.28	+10.88	+ .45	+ .94	
≥ 47	.0930	− .75	+ 63.60	+14.26	+ 8.94	+37.58	+17.03	+1.26	+2.28	
≥ 49	.0423	− 2.21	+118.60	+16.04	+14.86	+51.80	+22.19	+2.12	+3.75	
$N = 100$	$P = .45$									
≤ 30	.0015	+15.94	+655.15	−281.11	+36.22	−26.88	−18.05	+4.61	− .29	
≤ 33	.0098	+ 6.56	+292.82	−35.27	+23.76	−26.33	−16.98	+3.17	+ .23	
≤ 36	.0429	+ 2.02	+131.80	+12.73	+14.15	−24.90	−15.37	+1.98	+ .33	
≤ 39	.1343	+ .16	+ 55.21	+13.06	+ 7.11	−22.49	−13.24	+1.04	+ .16	
≤ 42	.3087	− .32	+ 17.53	+ 5.41	+ 2.37	−19.09	−10.65	+ .36	− .10	
≥ 46	.4587	+ .28	+ .39	+ .39	+ .39	+17.52	+ 9.19	+ .06	+ .39	
≥ 49	.2404	+ .18	+ 22.56	+ 7.93	+ 4.27	+32.14	+15.71	+ .68	+1.29	
≥ 52	.0960	− .28	+ 72.70	+19.21	+11.86	+54.99	+24.73	+1.93	+3.16	+1.47
≥ 54	.0441	− .77	+137.78	+23.80	+19.85	+76.94	+32.45	+3.24	+5.20	+2.19
≥ 57	.0106	− 1.62	+347.03	−11.27	+38.28	+125.49	+47.26	+6.18	+10.00	+3.10
≥ 60	.0018	− 2.18	+924.69	−312.57	+68.75	+205.01	+67.22	+10.75	+17.81	+3.62

Table 12. Relative errors in percentages of the binomial tail, cf. (2.2)

Region	Tail	Normal (3.8)	Poisson (3.1)	GR.CH (3.2)	Bolshev (3.5)	Wise (3.6)	Wimol (6.2)	Mep (6.4)	Bomol (4.1)	RVBML (4.2)
$N = 100$	$P = .50$									
≤ 35	.0018	+ 6.08	+821.86	−336.39	+50.11	−37.42	−25.42	+ 7.11	+ 1.16	
≤ 38	.0105	+ 2.24	+351.61	− 32.09	+32.34	−36.35	−23.78	+ 4.89	+ 1.31	
≤ 41	.0443	+ .57	+153.40	+ 20.27	+19.03	−34.25	−21.49	+ 3.05	+ 1.03	
≤ 44	.1356	+ .03	+ 62.98	+ 17.36	+ 9.48	−30.99	−18.54	+ 1.60	+ .52	
≤ 47	.3086	− .04	+ 19.77	+ 6.89	+ 3.16	−26.50	−15.00	+ .56	− .02	
≥ 51	.4602	− .01	+ .49	+ .49	+ .49	+24.49	+12.98	+ .09	+ .49	− .31
≥ 54	.2421	− .04	+ 25.62	+ 10.10	+ 5.61	+45.71	+22.38	+ 1.03	+ 1.78	+ .15
≥ 57	.0967	+ .13	+ 84.00	+ 25.59	+15.74	+80.39	+35.74	+ 2.93	+ 4.43	+ .70
≥ 59	.0443	+ .57	+162.66	+ 34.17	+26.64	+115.29	+47.49	+ 4.92	+ 7.32	+1.03
≥ 62	.0105	+ 2.24	+430.83	− 1.21	+52.68	+197.46	+70.86	+ 9.45	+14.15	+1.31
≥ 65	.0018	+ 6.08	+1241.99	−397.79	+98.19	+344.77	+104.01	+16.61	+25.49	+1.16
$N = 300$	$P = .01$									
≤ 0	.0490	+49.75	+ 1.52	− .00	+ .00	− .00	− .00	+ .00	− .00	
≤ 1	.1976	− 2.84	+ .76	+ .00	+ .00	− .00	− .00	+ .00	− .00	
≤ 2	.4221	− 8.58	+ .27	+ .00	+ .00	− .00	− .00	+ .00	− .00	
≥ 4	.3528	+ 9.38	+ .00	+ .00	+ .00	+ .00	+ .00	+ .00	+ .00	
≥ 5	.1839	+ 4.43	+ .46	+ .00	+ .00	+ .00	+ .00	+ .00	+ .00	
≥ 6	.0829	−11.42	+ 1.22	+ .01	+ .01	+ .01	+ .00	+ .00	+ .00	
≥ 7	.0328	−35.48	+ 2.31	+ .00	+ .01	+ .01	+ .01	+ .00	+ .00	
≥ 8	.0115	−60.68	+ 3.75	− .01	+ .02	+ .02	+ .01	+ .00	+ .01	
≥ 9	.0036	−80.35	+ 5.56	− .06	+ .03	+ .03	+ .01	+ .00	+ .01	
≥ 10	.0010	−92.06	+ 7.76	− .16	+ .04	+ .04	+ .02	+ .00	+ .02	
$N = 300$	$P = .10$									
≤ 15	.0013	+107.72	+53.76	−7.04	+ .96	− .58	− .40	+ .03	− .20	
≤ 18	.0097	+38.41	+33.15	−1.83	+ .66	− .62	− .40	+ .02	− .11	
≤ 21	.0458	+11.20	+18.86	+ .03	+ .40	− .61	− .37	+ .01	− .06	
≤ 24	.1439	+ .70	+ 9.26	+ .38	+ .21	− .57	− .33	+ .01	− .03	
≤ 27	.3224	− 2.22	+ 3.25	+ .20	+ .07	− .49	− .27	+ .00	− .02	
≥ 32	.3775	+ 2.36	+ 1.02	+ .09	+ .05	+ .56	+ .28	+ .00	+ .02	
≥ 35	.1914	+ .99	+ 5.93	+ .40	+ .19	+ .93	+ .45	+ .01	+ .04	
≥ 38	.0779	− 4.39	+14.31	+ .55	+ .44	+1.43	+ .66	+ .01	+ .08	
≥ 41	.0254	−14.71	+27.26	− .20	+ .80	+2.06	+ .90	+ .03	+ .15	
≥ 44	.0066	−29.50	+46.43	−3.28	+1.30	+2.84	+1.19	+ .04	+ .27	
≥ 47	.0014	−46.79	+74.29	−11.48	+1.96	+3.78	+1.51	+ .07	+ .44	

Table 13. Relative errors in percentages of the binomial tail, cf. (2.2)

Region	Tail	Normal (3.8)	Poisson (3.1)	GR.CH (3.2)	Bolshev (3.5)	Wise (3.6)	Wimol (6.2)	Mep (6.4)	Bomol (4.1)	RVBML (4.2)
N = 300	P = .30									
≤ 66	.0012	+29.04	+315.14	−120.04	+13.61	−17.72	−10.95	+1.10	− .42	
≤ 70	.0061	+14.71	+178.91	− 31.51	+ 9.76	−16.82	−10.10	+ .81	− .18	
≤ 74	.0239	+ 6.43	+100.23	− 2.10	+ 6.56	−15.58	− 9.08	+ .55	− .06	
≤ 78	.0723	+ 1.95	+ 53.50	+ 4.89	+ 4.01	−14.01	− 7.92	+ .35	− .01	
≤ 82	.1726	− .13	+ 25.45	+ 4.24	+ 2.07	−12.11	− 6.63	+ .18	− .02	
≤ 86	.3321	− .75	+ 8.94	+ 1.84	+ .73	− 9.96	− 5.27	+ .07	− .05	
≥ 91	.4716	+ .71	+ .09	+ .09	+ .09	+ 8.56	+ 4.37	+ .01	+ .09	
≥ 95	.2835	+ .67	+ 10.35	+ 2.37	+ 1.12	+13.53	+ 6.65	+ .10	+ .25	
≥ 99	.1423	− .17	+ 29.36	+ 5.33	+ 2.89	+20.06	+ 9.45	+ .27	+ .53	
≥ 103	.0590	− 2.23	+ 62.46	+ 6.48	+ 5.58	+28.39	+12.79	+ .53	+ .99	
≥ 107	.0200	− 5.86	+119.73	− 1.42	+ 9.38	+38.88	+16.70	+ .90	+1.69	
≥ 111	.0055	−11.24	+221.38	− 38.25	+14.50	+52.04	+21.21	+1.40	+2.70	
≥ 115	.0012	−18.35	+409.92	−158.74	+21.26	+68.54	+26.39	+2.06	+4.08	
N = 300	P = .40									
≤ 94	.0012	+14.11	+599.70	−274.87	+30.43	−39.94	−25.59	+3.33	+ .71	
≤ 100	.0102	+ 5.49	+239.32	− 27.07	+18,40	−36.82	−22.69	+2.13	+ .70	
≤ 106	.0550	+ 1.39	+ 94.92	+ 10.77	+ 9.70	−32.53	−19.18	+1.18	+ .47	
≤ 112	.1886	− .13	+ 32.19	+ 7.86	+ 3.86	−27.00	−15.12	+ .49	+ .16	
≤ 118	.4314	− .35	+ 4.66	+ 1.31	+ .50	−20.37	−10.74	+ .06	− .09	
≥ 125	.2970	+ .32	+ 13.13	+ 4.10	+ 2.05	+31.33	+15.41	+ .27	+ .50	
≥ 131	.1083	− .32	+ 55.59	+ 12.55	+ 7.46	+56.54	+25.60	+ .99	+1.51	
≥ 136	.0345	− 1.72	+133.89	+ 12.55	+15.15	+87.78	+36.64	+2.01	+3.01	
≥ 141	.0082	− 4.14	+304.95	− 33.82	+26.81	+132.82	+50.45	+3.50	+5.34	
≥ 146	.0014	− 7.66	+714.59	−289.49	+43.99	+198.78	+67.64	+5.57	+8.77	
N = 300	P = .50									
≤ 123	.0011	+ 2.43	+1126.93	−556.75	+63.29	−67.97	−47.72	+8.66	+4.44	
≤ 128	.0065	+ 1.01	+473.69	− 75.54	+41.10	−64.78	−43.99	+6.05	+3.47	
≤ 133	.0283	+ .32	+207.73	+ 16.99	+24.87	−60.58	−39.53	+3.90	+2.41	
≤ 138	.0921	+ .05	+ 89.26	+ 21.73	+13.34	−55.12	−34.32	+2.20	+1.39	
≤ 143	.2265	− .01	+ 32.99	+ 11.17	+ 5.58	−48.20	−28.36	+ .96	+ .52	
≤ 148	.4313	− .00	+ 5.87	+ 2.12	+ .90	−39.73	−21.84	+ .16	− .09	
≥ 154	.3431	− .01	+ 11.56	+ 4.72	+ 2.54	+57.43	+29.09	+ .45	+ .77	+ .12
≥ 159	.1632	− .00	+ 48.07	+ 16.54	+ 9.20	+102.14	+46.86	+1.64	+2.23	+ .84
≥ 164	.0594	+ .12	+128.32	+ 29.58	+20.81	+175.23	+71.30	+3.65	+4.79	+1.78
≥ 168	.0216	+ .42	+263.70	+ 23.49	+35.17	+267.46	+97.10	+6.00	+7.90	+2.62
≥ 172	.0065	+ 1.01	+548.86	− 59.08	+56.05	+409.60	+130.20	+9.17	+12.25	+3.47
≥ 177	.0011	+ 2.43	+1483.19	−658.20	+95.96	+710.48	+185.48	+14.56	+19.97	+4.44